Hofmann Revisited

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Contents

1 Preliminaries

The goal of this note is to give detailed proofs to certain parts of Uemura's work on representable map categories (particulary Theorem 5.17 in [Uem23]). The proof of Theorem 5.17 defers almost all of the lifting to a paper of Hofmann [Hof95], which itself leaves a good deal to the reader. We modernize Hofmann's setting and arguments to directly apply to the situation of Theorem 5.17, filling in missing details along the way.

Sections 1, 2, and 3 are all reasonably complete.

Definition 1.1. Let $\mathcal{C}$ be a cartesian category (closed under finite limits). A *stable class of representable arrows in* $\mathcal{C}$ is a class $R \subseteq \mathsf{Mor}\,\mathcal{C}$ such that

1. R contains all identities and is closed under composition (wide subcategory);
2. R is stable under pullback; and
3. Arrows in R are exponentiable (if $f \in R$ then f^* has a right adjoint). ♦

Definition 1.2. A *representable map category (RMC)* is a category $\mathcal{C}$ and a stable class of representable arrows R in $\mathcal{C}$.

A *representable map functor (RMF)* between RMCs $\mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ preserving finite limits, representable arrows, and pushforwards along representable arrows. ♦

Lemma 1.3. Given a discrete fibration $p : U \to \mathcal{C}$ and $A \in U$, we have an isomorphism of categories

$$U/A \cong \mathcal{C}/p(A) \qquad ♦$$

Proof. The functor $p : U/A \to \mathcal{C}/p(A)$ is our candidate isomorphism. Define $\ell : \mathcal{C}/p(A) \to U/A$ via unique lifting: given a map $f : \Gamma \to p(A)$ we can lift this to $\ell(f) : \ell(\Gamma) \to p$; by uniqueness of lifting this definition of $\ell(\Gamma)$ is well-defined, and likewise is functorial. We have $p \circ f = \mathrm{id}$ by construction and $f \circ p = \mathrm{id}$ by uniqueness of lifting. □

Lemma 1.4. Given a diagram $d : J \to \mathcal{C}$ where J has a terminal object 0 and a discrete fibration $p : U \to \mathcal{C}$, the set of factorizations of d through U is in bijection with $U(d(0))$.

Furthermore, p creates all limits of shape J. ♦

Proof. Fix $A \in U(d(0))$. Define $\hat{d} : J \to U$ on an object $n \in J$ as the domain of the unique lift of the map $d(n \to 0)$ with codomain A. On a morphism $i : n \to m \in J$, define $\hat{d}(i)$ to be the unique lift of $d(i)$ ending at $\hat{d}(m)$. A priori, the domain of this may not be what we wish; however, uniqueness forces this to be so:

$$\begin{tikzcd}
& \hat{d}(n) \arrow[drr] & & \\
? \arrow[r, "\hat{d}(i)"] & \hat{d}(m) \arrow[rr] & & A
\end{tikzcd}$$

Both $\hat{d}(n) \to A$ and $? \to A$ are lifts of $d(n \to 0)$ and so must be the same morphism. Identically, we obtain functoriality of $\hat{d}$.

Conversely, if we have some factorization $\hat{d}$ of d through U, then this factorization gives lifts of all the maps described by d and so must agree with the above construction.

Let $d : J \to U$ be a diagram. Since J has a terminal object, any cone over d is also a cone over $d : J \to U/d(0)$; furthermore, the projection $U/d(0) \to U$ will preserve and reflect the limit of d, if it exists. The same argument applied to $p \circ d : J \to \mathcal{C}$ and $\mathcal{C}/p(d(0))$ combined with Lemma 1.3 concludes the argument. $\square$

2 Hofmann's Splitting Technique

The goal of this section is to present the "splitting technique" due to Hofmann [Hof95] in a modern way. Throughout, fix a category $\mathcal{C}$ closed under pullbacks.

When working with categorical logic, substitution corresponds to taking pullbacks. The difficulty is that coming up with a strictly functorial choice of pullbacks is rather challenging, yet necessary: type theory requires iterated substitution to give the same result regardless of the order.

This can be solved generically via 2-categorical nonsense, i.e. by strictifying a certain 2-functor; such strictification is always possible. However, fixing pullbacks is only the first step in modelling type theory: we must construct further structure upon these "fixed" pullbacks. Hofmann's splitting technique provides a solution for this: a concrete strictification suitable for building type theory on.

Given $f : A \to B \in \mathcal{C}$, we can take the pullback functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$:

$$\begin{tikzcd} f^*C \arrow[r] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

One wishes to construct a 2-functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$ via this and the slice construction, i.e.:

$$\begin{aligned} A &\mapsto \mathcal{C}/A \\ f &\mapsto f^* \end{aligned}$$

The idea is that types over an object (context) $A \in \mathcal{C}$ will be precisely the maps into A, i.e. the objects of $\mathcal{C}/A$, and substitution/context extension is given by pullbacks.

As mentioned above, this is not in general going to be a strict 2-functor: only if the choice of pullbacks $f \mapsto f^*$ is functorial, which is unlikely. This does, however, describe a pseudofunctor

$$\mathcal{C}/- : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$$

as functoriality will hold up to natural equivalence by uniqueness of pullbacks.

The problem with strict functoriality is the uniqueness criterion we use: pullbacks are unique up to unique isomorphism. Hofmann's approach is to use the other uniqueness property from pullbacks: maps into a pullback are actually unique.

Definition 2.1. Let $A \in \mathcal{C}$. The *set of families over* A, denoted $\mathsf{Fam}(A)$, consists of the functors $\sigma : \mathcal{C}/A \to \mathcal{C}[1]$ sending morphisms in $\mathcal{C}/A$ to pullback squares (in $\mathcal{C}$) and fitting into:

$$\begin{tikzcd} \mathcal{C}/A \arrow[rr, "\sigma"] \arrow[dr, "\mathsf{dom}"'] & & \mathcal{C}[1] \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

Given a family $\sigma \in \mathsf{Fam}(A)$, the *canonical projection* of σ, denoted $p_A(\sigma)$ (or $p(\sigma)$ where A is understood), is the morphism

$$p(\sigma) := \sigma(\mathrm{id}_A) : A.\sigma \to A$$

where $A.\sigma := \mathsf{dom}\,\sigma(\mathrm{id}_A)$. ◆

Now we instead take "types over A" to be families over A: instead of a type being just a morphism into A, a type now must include a functorial choice of pullbacks.

It is important to note that, given a type $f : B \to A$ and a family $\sigma \in \mathrm{Fam}(A)$, we get a pullback square involving $\sigma(f)$ via the morphism part of σ by viewing f as a morphism from f to id_A in $\mathcal{C}/A$, i.e.:

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr, "f"'] & & A \arrow[dl, equal] \\
& A &
\end{tikzcd}$$

so applying σ to the *morphism* f:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\sigma(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & A.\sigma \arrow[d, "\sigma(\mathrm{id}_A)=p(\sigma)"] \\
B \arrow[r, "f"] & A
\end{tikzcd}$$

Via the canonical projection, we are on the way towards constructing an equivalence $\mathsf{Fam}(A) \simeq \mathcal{C}/A$, ignoring for now that the domain is not yet a category. The following construction helps us in the other direction:

Construction 2.2 (The hat construction). Fix any choice of pullbacks. Given a morphism $f : B \to A \in \mathcal{C}$, we wish to construct a family $\hat{f} : \mathcal{C}/A \to \mathcal{C}[1]$ such that $p(\hat{f}) \cong f$ in $\mathcal{C}/A$.

On an object $g : C \to A$, we define this as the pullback:

$$\begin{tikzcd}
\hat{f}(C) \arrow[r] \arrow[d, "\hat{f}(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}$$

Given a morphism

$$\begin{tikzcd}
C' \arrow[rr, "\alpha"] \arrow[dr, "h"'] & & C \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

we define this as the (unique) morphism into the pullback:

$$\begin{tikzcd}
\hat{f}(C') \arrow[rr] \arrow[dr, dashed, "\exists!\hat{f}(\alpha)"] \arrow[dd, "\hat{f}(h)"] & & B \arrow[dd, "f"] \\
& \hat{f}(C) \arrow[ur] \arrow[dd, "\hat{f}(g)", near start] & \\
C' \arrow[rr, "h", near start] \arrow[dr, "\alpha"'] & & A \\
& C \arrow[ur, "g"'] &
\end{tikzcd}$$

By the universal property of pullbacks, this guarantees that $\hat{f}$ will be functorial.

We compute $p(\hat{f})$ as the pullback

$$\begin{tikzcd}
\hat{f}(B) \arrow[r] \arrow[d, "p(\hat{f})=\hat{f}(\mathrm{id}_A)"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

Since the top is an isomorphism, we have $p(\hat{f}) \cong f$ in $\mathcal{C}/A$ as required. ⧫

Using canonical projections, we can move $\mathsf{Fam}(A) \to \mathcal{C}/A$; using the hat construction, we can move $\mathcal{C}/A \to \mathsf{Fam}(A)$. Note that the hat construction requires a choice of pullbacks: each choice of pullbacks will result in a different (but canonically isomorphic) family. To establish an equivalence between the two, we define morphisms on $\mathsf{Fam}(A)$ exactly so we have an equivalence:

Definition 2.3. The *category of families over* A, denoted $\mathsf{Fam}(A)$, has

- Objects the families over A (i.e. the set $\mathsf{Fam}(A)$); and
- Morphisms $f : \sigma \to \tau$ the morphisms "downstairs", i.e. morphisms $f : p(\sigma) \to p(\tau) \in \mathcal{C}/A$:

$$\begin{tikzcd}
A.\sigma \arrow[rr, "f"] \arrow[dr, "p(\sigma)"] & & A.\tau \arrow[dl, "p(\tau)"] \\
& A &
\end{tikzcd}$$

Composition is given by composition in $\mathcal{C}/A$. ⧫

Lemma 2.4. The canonical projection extends to a functor $p : \mathsf{Fam}(A) \to \mathcal{C}/A$. Furthermore, p is an equivalence of categories. ⧫

Proof. Its action on morphisms is clear: morphisms $\sigma \to \tau$ in $\mathsf{Fam}(A)$ are exactly morphisms $p(\sigma) \to p(\tau) \in \mathcal{C}/A$.

By construction, p is fully faithful. Via the hat construction, we see that p is essentially surjective. Thus, p is an equivalence of categories. □

We have established the object part of a functor $\mathsf{Fam} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$. What remains is to define this on morphisms.

In this case, our choice is clear: fix $\sigma : \mathcal{C}/B \to \mathcal{C}[1]$. Given the situation:

$$\begin{tikzcd}
C' \arrow[rr, "\alpha"] \arrow[dr, "h"] & & C \arrow[dl, "g"] \\
& A \arrow[d, "f"] & \\
& B &
\end{tikzcd}$$

we set on objects

$$\mathsf{Fam}(f)(\sigma)(g) := \sigma(f \circ g : C \to B)$$

and on morphisms

$$\mathsf{Fam}(f)(\sigma)(\alpha) := \sigma(\alpha : f \circ g' \to f \circ g)$$

The result is a family since $\mathsf{dom}(\mathsf{Fam}(f)(\sigma)(\alpha)) = \mathsf{dom}\,\sigma(\alpha) = \alpha$.

Since we define this via composition, the result is strictly functorial:

Proposition 2.5. The category-of-families construction gives a strict 2-functor

$$\mathsf{Fam} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$$ ♦

Via p we see that Fam is a strictification of $\mathcal{C}/-$:

Theorem 2.6. Fixing a choice of pullbacks to define $\mathcal{C}/- : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$ we have a pseudo-natural equivalence $p : \mathsf{Fam} \Rightarrow \mathcal{C}/-$. ♦

Proof. Consider a morphism $f : B \to A \in \mathcal{C}^{\mathsf{op}}$ and a square:

$$\begin{tikzcd} \mathsf{Fam}(A) \arrow[r, "p_A"] \arrow[d, "\mathsf{Fam}(f)"'] & \mathcal{C}/A \arrow[d, "f^*"] \\ \mathsf{Fam}(B) \arrow[r, "p_B"'] & \mathcal{C}/B \end{tikzcd}$$

Fix $\sigma : \mathcal{C}/A \to \mathcal{C}[1]$. The left composite gives:

$$\begin{aligned} p_B(\mathrm{Fam}(f)(\sigma)) &= (\mathrm{Fam}(f)(\sigma))(\mathrm{id}_B) \\ &= \sigma(f) \end{aligned}$$

Since σ is a family, this describes a pullback square:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\sigma(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & A.\sigma \arrow[d, "p(\sigma)"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

The right composite is exactly:

$$\begin{aligned} f^*(p_A(\sigma)) &= f^*\sigma(\mathrm{id}_A) \\ &= f^*p(\sigma) \end{aligned}$$

which is the pullback square:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f^*p(\sigma)"'] \arrow[dr, phantom, "\lrcorner", very near start] & A.\sigma \arrow[d, "p(\sigma)"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

In particular, we have a canonical isomorphism over B:

$$\begin{tikzcd} \cdot \arrow[rr, "\cong"] \arrow[dr, "\sigma(f)"'] & & \cdot \arrow[dl, "f^*p(\sigma)"] \\ & B & \end{tikzcd}$$

Since this is canonical, we have a natural isomorphism $p_B \circ \mathrm{Fam}(f) \cong f^* \circ p_A$. Again by canonicity, the other axioms of a pseudonatural equivalence are satisfied. □

3 Natural Models of Type Theory

This section is mainly focused on relating natural models to the more traditional definitions of models of type theory: namely, categories with attributes and their ilk. Many of the arguments in [Hof95] are most naturally applied in this fashion.

Definition 3.1. A *natural model* is a category $\mathcal{C}$ with terminal object and a representable map $\partial : E \to U$ of discrete fibrations. ◆

Fix a natural model

$$\begin{tikzcd} E \arrow[rr, "\partial"] \arrow[dr, two heads] & & U \arrow[dl, two heads] \\ & \mathcal{C} & \end{tikzcd}$$

The intuition is that $\Gamma \in \mathcal{C}$ correspond to contexts in our model, objects $A \in U(\Gamma)$ correspond to types in context Γ, and objects $a \in E(\Gamma)$ correspond to terms in context Γ whose type is given by $\partial a \in U(\Gamma)$.

Given a morphism $f : \Delta \to \Gamma$ of contexts, by the unique lifting properties of the fibrations we have substitution operations on types $A \in U(\Gamma)$ giving $f^*A \in U(\Delta)$ and on terms $a \in E(\Gamma)$ giving $f^*a \in E(\Delta)$. By uniqueness of lifts and commutativity of the triangle, we have $\partial(f^*a) = f^*A$. Again by uniqueness iterated substitution may be performed in any order.

Since ∂ is asked to be representable, we may choose a right adjoint $d : U \to E$ for it (generally not a map of discrete fibrations) which comes with unit $\eta : \mathrm{id} \Rightarrow d\partial$ and counit $\varepsilon : \partial d \Rightarrow \mathrm{id}$. Fix a type $A \in U(\Gamma)$. Via the counit we have a map

$$\varepsilon_A : \partial dA \to A$$

Applying p_U, we have a map

$$p_A := p_U \varepsilon_A : p(\partial dA) \to \Gamma$$

We define $\Gamma.A := p(\partial dA)$ to be context extension. Our choice of right adjoint necessarily affects context extension; regardless, context extension is unique up to isomorphism by uniqueness of adjoints.

Lemma 3.2. Context extension gives a functor $e : U \to \mathcal{C}$ and projections a natural transformation $p : e \Rightarrow p_U$. Given a map $f : \Gamma \to \Delta \in \mathcal{C}$ and $A \in U(\Gamma)$, we have a functorial (in f) choice of $q(A, f)$ making the following a pullback square:

$$\begin{tikzcd} \Delta.f^*A \arrow[r, "q(A,f)"] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

◆

Proof. We define $e := p_U \circ \partial \circ d$ and $p := p_U \circ \varepsilon$. Note that $\partial \circ d$ and ε are the functor and counit of a comonad.

Given $A \in U(\Gamma)$ and $f : \Delta \to \Gamma$, take the unique lift of f to get $\hat{f} : f^*A \to A \in U$; set $q(A, f) := e(\hat{f})$. That the square commutes is exactly naturality of $p : e \Rightarrow p_U$. Functoriality of q follows from its definition as a unique lift followed by a functor.

All that remains is to show the above is a pullback square. We have yet to impose any requirement for $\mathcal{C}$ to have pullbacks at all, so we must prove the universal property directly. Suppose we have the situation

$$\begin{tikzcd}
B \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& \Delta.f^*A \arrow[r, "q(A,f)"] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"] \\
& \Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

We have a lift of the map g to U, call it $\hat{g} : (fh)^*A \to \partial dA$, and a lift of g to E, call if $\bar{g} : \bar{B} \to dA$, with $\partial\bar{g} = \hat{g}$ by uniqueness of lifts. Thus, $(fh)^*A = \partial\hat{B}$; via the adjunction:

$$E(\bar{B}, df^*A) \cong U(\partial\bar{B}, f^*A) = U((fh)^*A, f^*A)$$

Via unique lifting we have a map $\hat{h} : (fh)^*A \to f^*A$; thus, we get $\bar{h} : \bar{B} \to df^*A$ via the adjunction. Since the diagram category of this shape has a terminal object, by Lemma 1.4 we can lift the thing wholesale into U and show it is a pullback there:

$$\begin{tikzcd}
(fh)^*A \arrow[drr, bend left, "\hat{g}=\partial\bar{g}"] \arrow[dr, "\partial\bar{h}"] \arrow[ddr, bend right, "\hat{h}"'] & & \\
& \partial df^*A \arrow[r, "\partial d\hat{f}"] \arrow[d, "\varepsilon_{f^*A}"] & \partial dA \arrow[d, "\varepsilon_A"] \\
& f^*A \arrow[r, "\hat{f}"] & A
\end{tikzcd}$$

The bottom triangle commutes since $\bar{h}$ is the adjoint transpose of $\hat{h}$. For the top, we must show the following commutes in E:

$$\begin{tikzcd}
\bar{B} \arrow[drr, bend left, "\bar{g}"] \arrow[dr, "\bar{h}"] & & \\
& df^*A \arrow[r, "d\hat{f}"] & dA
\end{tikzcd}$$

We have

$$\bar{h} = \bar{B} \xrightarrow{\eta_{\bar{B}}} d\partial\bar{B} \xrightarrow{d(\hat{h})} df^*A$$

So

$$\begin{aligned}
d\hat{f} \circ \bar{h} &= d\hat{f} \circ d\hat{h} \circ \eta_B \\
&= d(\hat{f} \circ \hat{h}) \circ \eta_B \\
&= d(\varepsilon_A \circ \hat{g}) \circ \eta_B \\
&= d(\varepsilon \circ \partial\bar{g}) \circ \eta_B
\end{aligned}$$

Now this is the transpose of $\varepsilon \circ \partial\bar{g}$, which is the transpose of $\bar{g}$; double transpose does nothing, so this is equal to $\bar{g}$.

Suppose we have another candidate $\ell : (fh)^*A \to \partial df^*A$. This likewise comes from a map $\bar{\ell} \in E$, i.e. $\ell = \partial\bar{\ell}$. Then commutativity exhibits $\bar{\ell}$ as the adjoint transpose of $\bar{h}$, which is unique; thus, we have a pullback square in U. By Lemma 1.4, the map p_U preserves pullback squares and so our square in $\mathcal{C}$ is a pullback. $\square$

In any definition of a model of type theory where we have a projection map $\Gamma.A \to \Gamma$ corresponding to context extension by a type A, there will be some relationship between sections of $\Gamma.A \to \Gamma$ and whatever the definition gives as a "term of type A in context Γ".

Lemma 3.3. Given $\Gamma \in \mathcal{C}$ and $A \in U(\Gamma)$, we have a bijection

$$\{a \in E(\Gamma) \text{ with } \partial a = A\} \leftrightsquigarrow \{\text{sections of } \Gamma.A \to \Gamma\}$$ ♦

Proof. Suppose we have a term $a \in E(\Gamma)$. We claim that this defines a section of the projection map $p_A : \Gamma.A \to \Gamma$. Via the unit we have

$$\eta_a : a \to d\partial a$$

Applying p_E we have a map $p_E(\eta_a) : \Gamma \to p_E(d\partial a)$. We have

$$p_E(d\partial a) = p_U(\partial d\partial a) = p_U(\partial dA) = \Gamma.A$$

Call this map $\hat{a} : \Gamma \to \Gamma.A$. This is the desired section:

$$\begin{aligned} p_A \circ \hat{a} &= p_U(\varepsilon_{\partial a}) \circ p_E(\eta_a) \\ &= p_U(\varepsilon_{\partial a}) \circ p_U(\partial \eta_a) \\ &= p_U(\varepsilon_{\partial a} \circ \partial \eta_a) \\ &= p_U(\mathrm{id}) \\ &= \mathrm{id} \end{aligned}$$

by a triangle identity for $\partial \dashv d$. Thus, given a term $a : A$ we get a section of $\Gamma.A \to \Gamma$ as $\hat{a} := p_E(\eta_a)$.

Conversely, suppose we have a section $a : \Gamma \to \Gamma.A$. We have the term $dA \in E(\Gamma.A)$, so by unique lifting we obtain a lift $\bar{a} \to dA$; the term corresponding to a is the domain, $\bar{a}$.

Now let $a \in E(\Gamma)$. Our section is exactly $p_E(\eta_a)$; in particular, η_a is a lift of the section. Thus, by uniqueness of lifting the term corresponding to $p_E(\eta_a)$ must be a.

Finally, let $a : \Gamma \to \Gamma.A$ be a section. We have that:

$$E(\bar{a}, dA) \cong U(A, A)$$

The transpose of the lift of our section $\bar{a} : \bar{a} \to dA$ is:

$$\partial\bar{a} = A \xrightarrow{\partial\bar{a}} \partial dA \xrightarrow{\varepsilon_A} A$$

Yet since $\bar{a}$ is a lift to E of a, $\partial\bar{a}$ is a lift to U of a and so is a section of $\varepsilon_{\partial a}$; thus, $\bar{a}$ is the adjoint transpose of $\mathrm{id}_{\partial a}$ and so $\bar{a} = \eta_a$. $\square$

Definitions of models of type theory can be split into two classes: those which are "termed" and those which are not. A definition is "termed" if it gives a definition for "term of type A in context Γ" other than "section of $\Gamma.A \to \Gamma$." Lemma 3.3 tells us that while natural models appear to be "termed" models by the presence of E in the definition, they are actually not: at least the objects of E can be entirely recovered from sections of the $\Gamma.A \to \Gamma$ (given a choice of right adjoint d), and we will see that this is true for the entire category E.

Despite this, it is morally helpful to consider E to be the category of terms in our model; however, the actual value of E is, with a choice of right adjoint $d : U \to E$, to give a functorial construction of context extension and the projections $\Gamma.A \to \Gamma$.

Definition 3.4. Given a natural model and right adjoint $d : U \to E$ of $\partial : E \to U$, form the *category of sections*, denoted $\mathsf{Sec}(U)$, with:

- Objects the triples $(\Gamma \in \mathcal{C}, A \in U(\Gamma), a : \Gamma \to \Gamma.A)$ for a a section of $p_A : \Gamma.A \to \Gamma$; and
- Morphisms $(\Gamma, A, a) \to (\Delta, B, b)$ the morphisms $f : \Gamma \to \Delta$ in $\mathcal{C}$ such that $B = f^*A$ and $q(A, f) \circ b = a \circ f$, i.e. the following commutes:

$$\begin{tikzcd}
\Gamma.A \arrow[r, "q(A,f)"] & \Delta.B \\
\Gamma \arrow[u, "a"] \arrow[r, "f"] & \Delta \arrow[u, "b"']
\end{tikzcd}$$

♦

Lemma 3.5. The obvious projection $p : \mathsf{Sec}(U) \to \mathcal{C}$ is a discrete fibration. ♦

Proof. Since morphisms in $\mathsf{Sec}(U)$ are exactly morphisms downstairs (subject to constraints), lifts are unique if they exist: given a section $a : \Gamma \to \Gamma.A$ and a morphism $f : \Delta \to \Gamma$ we must come up with a unique domain (Δ, f^*A, b). We have the pullback square by Lemma 3.2:

$$\begin{tikzcd}
\Delta.f^*A \arrow[r, "q(A,f)"] \arrow[d, "p_{f^*A}"] \arrow[dr, phantom, "\lrcorner"] & \Gamma.A \arrow[d, "p_A"'] \\
\Delta \arrow[r, "f"] & \Gamma \arrow[u, bend right, "a"']
\end{tikzcd}$$

Via the standard method:

$$\begin{tikzcd}
\Delta \arrow[dr, dashed, "\exists! b"] \arrow[drr, bend left, "a \circ f"] \arrow[ddr, equal, bend right] & & \\
& \Delta.f^*A \arrow[r, "q(A,f)"] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"] \\
& \Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

we get the section b of $\Delta \to \Delta.f^*A$. By the universal property of pullbacks, this section is unique so we have a unique lift of f. □

Lemma 3.6. The correspondence of Lemma 3.3 extends to give an isomorphism of categories over $\mathcal{C}$:

$$F : E \cong \mathsf{Sec}(U) : G$$

Furthermore, there is a canonical map $\delta : \mathsf{Sec}(U) \to U$ over $\mathcal{C}$ making the following commute:

$$\begin{tikzcd}
E \arrow[rr, "F"] \arrow[dr, "\partial"] & & \mathsf{Sec}(U) \arrow[dl, "\delta"] \\
& U &
\end{tikzcd}$$

♦

Proof. Define $F : E \to \mathsf{Sec}(U)$ on objects as:

$$F(a \in E(\Gamma)) := (\Gamma, \partial a, p_E(\eta_a))$$

and $G : \mathsf{Sec}(U) \to E$ on a section $a : \Gamma \to \Gamma.A$ as the domain of the unique lift $\bar{a} \to dA$. By Lemma 3.3, this is an isomorphism on objects.

It remains to define both on morphisms. Given a morphism $f : b \to a \in E$ (with $A := \partial a$, $B := \partial b$, $\Gamma := p_E(a)$, and $\Delta := p_E(b)$), we must define $F(f) := p_E(f)$. We check that this is well-defined; by uniqueness of lifts, $B = (p_E(f))^* A$ since $\partial f : B \to A$ is a lift of $p_E(f)$. We claim that $q(A, p_E(f)) \circ p_E(\eta_b) = p_E(\eta_a) \circ f$; this is exactly naturality of $p_E \eta$. Thus, F is well-defined.

For G, we have no choice: we must set $G(f : (\Gamma, A, a) \to (\Delta, B, b))$ to be the unique lift of f ending at $G(b)$. By uniqueness of lifts, since we can lift the whole diagram showing that f is a morphism in $\mathsf{Sec}(U)$ up p_E via Lemma 1.4, this is a morphism $G(a) \to G(b)$.

By construction and uniqueness of lifts, $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.

The functor $\delta : \mathsf{Sec}(U) \to U$ is given on objects by projection onto the second coordinate; on morphisms it is given by unique lifting. □

We do not care at the moment to define morphisms of natural models; however, using whatever definition you please:

Corollary 3.7. Given a natural model $(\mathcal{C}, \partial : E \to U)$ and a choice of right adjoint $\partial \dashv d$, we have a natural model

$$\begin{tikzcd}
\mathsf{Sec}(U) \arrow[rr, "\delta"] \arrow[dr, two heads, "p"'] & & U \arrow[dl, two heads, "p_U"] \\
& \mathcal{C} &
\end{tikzcd}$$

isomorphic to the original. ♦

It is worth reiterating once again that the construction here depends on the specific right adjoint d.

References

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